$$x_1^2+x_2^2+x_3^2+3(x_1x_2+x_2x_3+x_3x_1)=0.$$

To this conic the triangle B_1, B_2, B_3 (or E_1, E_2, E_3) bears the same relation as the original triangle A to the original conic, *i. e.* the tangents to the conic at B_1, B_2, B_3 (or E_1, E_2, E_3) pass through C_1, C_2, C_3, respectively.

Likewise the conic

$$x_1^2+x_2^2+x_3^2-2(x_1x_2+x_2x_3+x_3x_1)=0$$

is inscribed in the two triangles A and D. The lines joining the contact points of these two triangles pass again through C_1, C_2, C_3. In this way we obtain an infinite chain of conics Γ. All these conics Γ pass through the same points ω_1, ω_2 on the line ω, and since the point O is the pole of ω with respect to every conic, the conics Γ are tangent to each other at ω_1 and ω_2.

If we transform by projection the original conic into a circle and simultaneously the line ω into the line at infinity, which is always possible, then the conics Γ are projected into conics passing through the two circular points at infinity where they are tangent to each other, *i. e.* into concentric circles. The inscribed and circumscribed triangles of the diagram are transformed into equilateral triangles.

From this system of concentric circles with inscribed and circumscribed triangle all the properties of the diagram involving the conics Γ can now easily be deduced.

If we take as original conic an ellipse and as line ω the line at infinity, the point O becomes the center of the ellipse and also the point of intersection of the medians of every triangle A_1, A_2, A_3, and it can easily be shown not only that all these triangles are of equal area, but also that this area represents the maximum value of the areas of all triangles inscribed into the ellipse.

THE UNIVERSITY OF CHICAGO, *November, 1905.*

A PERFECT MAGIC SQUARE.

By PROFESSOR F. ANDEREGG.

In the accompanying magic square, the sum of the numbers in any line, column, or diagonal is the same (2056). Instead of continuous lines, any four numbers can be taken from the first and third groups of four numbers, and any four from the second and fourth groups in the first line, and the corresponding numbers in the last line. Similar combinations can be made for the second and fifteenth lines, the third and fourteenth, etc. Similar combinations can also be made for the columns.

Instead of a complete diagonal two incomplete diagonals can be taken on

opposite sides of either complete diagonal so that the two together contain sixteen numbers.

The sum of the sixteen numbers in any four two-squares symmetrically situated with respect to the center of the sixteen-square is 2056. The sum of the sixty-four numbers in any eight-square taken at random, or of any four four-squares symmetrically situated with respect to the center of the sixteen-square is 8224.

1	2	3	4	248	247	246	245	73	74	75	76	192	191	190	189
17	18	19	20	232	231	230	229	89	90	91	92	176	175	174	173
33	34	35	36	216	215	214	213	105	106	107	108	160	159	158	157
49	50	51	52	200	199	198	197	121	122	123	124	144	143	142	141
128	127	126	125	137	138	139	140	56	55	54	53	193	194	195	196
112	111	110	109	153	154	155	156	40	39	38	37	209	210	211	212
96	95	94	93	169	170	171	172	24	23	22	21	225	226	227	228
80	79	78	77	185	186	187	188	8	7	6	5	241	242	243	244
181	182	183	184	68	67	66	65	253	254	255	256	12	11	10	9
165	166	167	168	84	83	82	81	237	238	239	240	28	27	26	25
149	150	151	152	100	99	98	97	221	222	223	224	44	43	42	41
133	134	135	136	116	115	114	113	205	206	207	208	60	59	58	57
204	203	202	201	61	62	63	64	132	131	130	129	117	118	119	120
220	219	218	217	45	46	47	48	148	147	146	145	101	102	103	104
236	235	234	233	29	30	31	32	164	163	162	161	85	86	87	88
252	251	250	249	13	14	15	16	180	179	178	177	69	70	71	72

OBERLIN COLLEGE, *April 26, 1905.*

DEPARTMENTS.

NOTE. All solutions of problems, problems for solution, and other department contributions should be sent direct to THE AMERICAN MATHEMATICAL MONTHLY, 1227 Clay Street, Springfield, Mo.

SOLUTIONS OF PROBLEMS.

The following problem was received too late for publication: Miscellaneous No. 150, solved by L. E. Newcomb.

ALGEBRA.

243. Proposed by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

Find the infinite root of $\frac{1}{x}+\frac{1}{a}=\sqrt{\left[\frac{1}{a^2}-\sqrt{\frac{1}{a^2x^2}+\frac{1}{x^4}}\right]}$.

Solution by ELMER SCHUYLER, A. B., Brooklyn, New York.

Squaring the sides and reducing we have,

$$\frac{1}{x^2}+\frac{2}{ax}=-\sqrt{\frac{1}{a^2x^2}+\frac{1}{x^4}}.$$

Whence $1/x=0$, or $x=\infty$. Also,

$$\frac{1}{x}+\frac{2}{a}=-\sqrt{\frac{1}{a^2}+\frac{1}{x^2}}.$$

Squaring this and reducing, we get

$$\frac{1}{x}=-\frac{3}{4a}, \text{ or } x=-\frac{4a}{3}.$$

Since $\frac{1}{x^2}$ cancelled out on each side of the equation, $\frac{1}{x}=0$, or $x=\infty$. The only value of x satisfying the original equation, the radicals being taken with positive signs, is $x=\infty$.

Also solved by B. F. Finkel, G. W. Greenwood, J. E. Sanders, G. B. M. Zerr, and the Proposer.

AVERAGE AND PROBABILITY.

168. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the average area of a triangle two of whose sides have the constant sum $2a$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let x, y be the sides and θ the included angle.

Then $\frac{1}{2}xy\sin\theta$=area, and $x+y=2a$.

$$\therefore \text{Average area}=\triangle=\frac{\frac{1}{2}\int_0^{2a}\int_0^{\pi}x(2a-x)\sin\theta\,dx\,d\theta}{\int_0^{2a}\int_0^{\pi}dx\,d\theta}.$$

$$\therefore \triangle=\frac{1}{4\pi a}\int_0^{2a}\int_0^{\pi}x(2a-x)\sin\theta\,dx\,d\theta=\frac{1}{2\pi a}\int_0^{2a}x(2a-x)dx=\frac{2a^2}{3\pi}.$$

169. Proposed by HENRY HEATON, Atlantic, Iowa.

What is the average length of all straight lines that can be drawn within a given square?

I. Solution by B. F. FINKEL, A. M., Fellow in The University of Pennsylvania.

Let $ABCD$ be the given square, the length of whose side is a; and let EF and $E'F'$ be two parallel lines: EF terminating in two adjacent sides of the square, and $E'F'$ in two opposite sides. Let $AH,=x$, be the perpendicular to EF from A, and let the angle $FAH=\theta$.

Then $EF=\frac{x}{\sin\theta\cos\theta}$, and $E'F'=\frac{a}{\cos\theta}$.

Since the law of distribution of the lines is not given, we will assume that those having a given direction are uniformly distributed, and those passing through a given point are distributed so that the number of lines lying in a given angle is directly proportional to the magnitude of the angle.

Now, the limits of x are 0 and $AH'=a\sin\theta$ for $EF=x/\sin\theta\cos\theta$, and $AH'=a\sin\theta$ and $AH''=a\cos\theta$, for $EF=E'F'=a/\cos\theta$; and the limits of θ are 0 and $\frac{1}{4}\pi$. Then the average length of lines drawn within the square is

$$l_{\triangle}=\frac{\int_0^{\frac{1}{4}\pi}d\theta\left[2\int_0^{a\sin\theta}\sec\theta\cos\theta\,xdx+\int_{a\sin\theta}^{a\cos\theta}a\sec\theta\,dx\right]}{\int_0^{\frac{1}{4}\pi}d\theta\int_0^{\sqrt{2}\,a\cos(\frac{1}{4}\pi-\theta)}dx}$$

$$=\frac{1}{a}\int_0^{\frac{1}{4}\pi}d\theta\left[2\int_0^{a\sin\theta}\sec\theta\csc\theta\,xdx+a\int_{a\sin\theta}^{a\cos\theta}\sec\theta\,dx\right]$$

$$=\frac{1}{a}\int_0^{\frac{1}{4}\pi}d\theta\left[\sec\theta\csc\theta\left(x^2\right)_0^{a\sin\theta}+a\sec\theta\left(x\right)_{a\sin\theta}^{a\cos\theta}\right]$$

$$=\frac{1}{a}\int_0^{\frac{1}{4}\pi}a^2\,d\theta=\tfrac{1}{4}\pi a.$$

II. Solution* by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $ABCD$ be the given square. Let F be a point in AB, E a point in AD, and G, H, points in DC. Let PQ be a length, measured at random in FE and FG. Let $AF=x$, $AE=y$, $PQ=z$, $DH=u$, $DG=v$, $AB=a$, $\sqrt{(x^2+y^2)}=z'$, $\sqrt{[a^2+(u-v)^2]}=z''$. M=the average length.

$$\therefore M=\frac{\int_0^a\int_0^a\int_0^{z'} z\,dx\,dy\,dz+\int_0^a\int_0^u\int_0^{z''} z\,du\,dv\,dz}{\int_0^a\int_0^a\int_0^{z'} dx\,dy\,dz+\int_0^a\int_0^u\int_0^{z''} du\,dv\,dz}$$

$$=\frac{\int_0^a\int_0^a(x^2+y^2)dx\,dy+\int_0^a\int_0^u[a^2+(u-v)^2]du\,dv}{2\int_0^a\int_0^a\sqrt{(x^2+y^2)}dx\,dy+2\int_0^a\int_0^u\sqrt{[a^2+(u-v)^2}du\,dv}$$

$$=\frac{\int_0^a a(3x^2+a^2)dx+\int_0^a(3a^2u+u^3)du}{3\int_0^a\left[a\sqrt{(a^2+x^2)}+x^2\log\left(\frac{a+\sqrt{(a^2+x^2)}}{x}\right)\right]dx+3\int_0^a\left[u\sqrt{(a^2+u^2)}+a^2\log\left(\frac{u+\sqrt{(a^2+u^2)}}{a}\right)\right]du}$$

$$=\frac{11a}{4[2+\sqrt{2}+5\log(1+\sqrt{2})]}.$$

Also solved by J. E. Sanders.

171. Proposed by O. E. GLENN, A. M., Ph. D., Drury College.

There are n derelict steamers afloat in a circular sea of radius r. The water in the sea is moving northward in a current whose velocity varies inversely as the perpendicular distance from the north-south tangent to the sea on its west beach. Find the probability that a ship crossing the sea on a random diameter will encounter e derelicts during the voyage.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let O be the center of the sea, POQ the random diameter, G the position of a derelict when the start is made at P, AOB the diameter perpendicular to the north-south tangent AC on the west beach A. Let FE be the chord through G perpendicular to AB, cutting AB in L and PQ in K, H the foot of the perpendicular from Q on AB. Let $\angle QOB=\theta$, $AL=z$, $GK=y$, u=uniform velocity of ship on PQ, p=chance of meeting one derelict. Then $OK=OL\sec\theta=(z-r)\sec\theta$, velocity of derelict$=m/z$.

*The two methods illustrated are typical of the different results to be obtained when different laws of distribution are assumed.

$$\therefore \frac{r+(z-r)\sec\theta}{u}=\frac{yz}{m}. \qquad \therefore y=\frac{m}{uz}(r-r\sec\theta+z\sec\theta)=y'.$$

The limits of θ are 0 and 2π; of z, 0 and $2r$ for total, and $r(1+\cos\theta)=z'$ to $r(1-\cos\theta)=z''$ for favorable cases; of y, 0 and $2\sqrt{(2rz-z^2)}=y''$ for total, and 0 and y' for favorable cases.

$$\therefore p=\frac{\int_0^{2\pi}\int_{z''}^{z'}\int_0^{y'}d\theta\,dz\,dy}{\int_0^{2\pi}\int_0^{2r}\int_0^{y''}d\theta\,dz\,dy}=\frac{1}{2\pi^2r^2}\int_0^{2\pi}\int_{z''}^{z'}\int_0^{y'}d\theta\,dz\,dy$$

$$=\frac{m}{2\pi^2r^2u}\int_0^{2\pi}\int_{z''}^{z'}\frac{(r-r\sec\theta+z\sec\theta)d\theta\,dz}{z}$$

$$=\frac{m}{\pi^2ru}\int_0^{2\pi}(1+\sec\theta\,\log\tan\tfrac{1}{2}\theta-\log\tan\tfrac{1}{2}\theta)d\theta=\frac{2m}{\pi ru}.$$

Therefore the chance of meeting one derelict out of the n is $\Sigma p=2nm/\pi ru$, and the probability that e will be encountered is

$$\Pi\left(\frac{2nm}{\pi ru}\right)=\left(\frac{2nm}{\pi ru}\right)^e.$$

CALCULUS.

203. Proposed by S. A. COREY, Hiteman, Iowa.

Evaluate* $\int_0^\pi \frac{\sin mx\,dx}{x}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\int_0^\pi \frac{\sin mx}{x}dx=\int_0^\pi\left(m-\frac{m^3x^2}{3!}+\frac{m^5x^4}{5!}-\frac{m^7x^6}{7!}+\ldots\ldots\right)dx$$

$$=m\pi\left(1-\frac{m^2\pi^2}{3.3!}+\frac{m^4\pi^4}{5.5!}-\frac{m^6\pi^6}{7.7!}+\ldots\ldots\right).$$

Also solved by L. E. Newcomb.

204. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Required the variation of $\int V dx$ where V is a function of x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ .... and v where $v=\int V'dx$ and V' is also a function of x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$....

*See Byerly's *Integral Calculus* (p. 23, Table of Integrals) Formulae 211 and 241. A solution not in the form of an infinite series would also be desirable.

*Solution by the PROPOSER.

The variation of $\int V dx$ may be divided into $\triangle u$ and $\triangle u_1$, the former arising supposing v constant, the latter from the variation of v. Thus $\triangle u = H + \int K \triangle y dx$, $\triangle u_1 = \int \frac{dV}{dv} \triangle v dx$. If the letters represent for V what their primes represent for V', $\triangle u = \int (N' \triangle y + P' \triangle p + \ldots\ldots) dx$. If $L = \frac{dV}{dv}$ and $I = \int L dx$,

$$\triangle u_1 = \int L \triangle v dx = Idv - \int I \frac{d \triangle v}{dx} dx = I(H' + \int K' \triangle y dx) - (Hi' + \int Ki' \triangle y dx)$$

(Hi, Ki denoting H' and K' when IN', IP', etc., are substituted for N', P', ... Then

$$\triangle u + \triangle u_1 = \triangle y \left(P - \frac{dQ}{dx} \ldots\right) + \triangle p \left(Q - \frac{dR}{dx} + \ldots\right) + \ldots\ldots$$

$$+ \int \left(N - \frac{dP}{dx} + \frac{d^2 Q}{dx^2} - \ldots\right) \triangle y dx + I \triangle y \left(P' - \frac{dQ'}{dx} + \frac{d^2 R'}{dx^2} \ldots\right)$$

$$+ I \triangle p \left(Q' - \frac{dR'}{dx} + \ldots\right) + \ldots + I \int \left(N' - \frac{dP'}{dx} + \frac{d^2 Q'}{dx^2} \ldots\right) dx$$

$$- \triangle y \left(IP' - \frac{dIQ'}{dx} + \ldots\right) - \triangle p \left(IP' - \frac{dIQ'}{dx} + \ldots\right) - \ldots\ldots$$

$$- \int \left(IN' - \frac{dIP'}{dx} + \frac{d^2 IQ'}{dx^2} \ldots\right) dx.$$

Also solved by G. B. M. Zerr.

205. Proposed by Z. T. JACKSON, St. Louis, Mo.

Evaluate† $\int_0^{\frac{1}{2}\pi} \log \sin x \, dx$.

Solution by J. E. SANDERS, Reinersville, Ohio.

$$u = \int_0^{\frac{1}{2}\pi} \log \sin x \, dx = \int_0^{\frac{1}{2}\pi} \log \sin(\tfrac{1}{2}\pi - x) dx = \int_0^{\frac{1}{2}\pi} \log \cos x \, dx.$$

$$\therefore 2u = \int_0^{\frac{1}{2}\pi} (\log \sin x + \log \cos x) dx = \int_0^{\frac{1}{2}\pi} \log(\sin x \cos x) dx$$

$$= \int_0^{\frac{1}{2}\pi} \log \frac{\sin 2x}{2} dx = \int_0^{\frac{1}{2}\pi} \log \sin 2x \, dx - \frac{\pi}{2} \log 2.$$

*See Williamson's *Integral Calculus*, Sixth Edition, p. 275.
†Byerly, *Integral Calculus*, p. 102.

Let $2x=x'$, then

$$\int_0^{\frac{1}{2}\pi}\log\sin 2x\,dx=\tfrac{1}{2}\int_0^{\pi}\log\sin x'\,dx'=\int_0^{\frac{1}{2}\pi}\log\sin x\,dx.$$

$$\therefore 2u=u-\frac{\pi}{2}\log 2,\quad u=-\frac{\pi}{2}\log 2=\frac{\pi}{2}\log\tfrac{1}{2}.$$

Also solved by M. E. Graber, G. W. Greenwood, L. E. Newcomb, and G. B. M. Zerr.

206. Proposed by DR. O. E. GLENN, Drury College.

Evaluate $\int_0^1(1-z^n)^m\frac{\partial}{\partial z}\log(1-z^nx^n)dz$, assuming $-1<x^n<+1$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$u=-\int_0^1\frac{nx^nz^{n-1}(1-z^n)^m}{1-x^nz^n}dz$. Let $1-z^n=y$, then we have

$$u=-\int_0^1\frac{x^ny^m}{1-x^n+x^ny}dy=-\int_0^1\frac{y^m}{y+a}dy,\text{ where }\frac{1-x^n}{x^n}=a.$$

$$\therefore u=-\int_0^1\left(y^{m-1}-ay^{m-2}+a^2y^{m-3}\ldots\ldots(-1)^{m-1}a^{m-1}+\frac{(-1)^ma^m}{y+a}\right)dy$$

$$=-\frac{y^m}{m}-\frac{ay^{m-1}}{m-1}+\ldots\ldots+(-1)^{m-1}a^{m-1}y+(-1)^ma^m\log(y+a)\Big]_0^1$$

$$=-\left[\frac{1}{m}-\frac{a}{m-1}+\ldots\ldots+(-1)^{m-1}a^{m-1}+(-1)^ma^m\log\left(\frac{1+a}{a}\right)\right]$$

$$=-\left[\frac{1}{m}-\frac{(1-x^n)}{x^n(m-1)}+\frac{(1-x^n)^2}{x^{2n}(m-2)}+\ldots\ldots\frac{(-1)^{m-1}(1-x^n)^{m-1}}{x^{(m-1)n}}\right.$$

$$\left.+(-1)^{m+1}\frac{(1-x^n)^m}{x^{mn}}\log(1-x^n)\right].$$

DIOPHANTINE ANALYSIS.

128. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Required the highest powers of 2, 3, 5, 7, contained in (1000)!

I. Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

$$(1000)!=2^{500}(500)!(1.3.5\ldots\ldots 999)$$
$$(500)!=2^{250}(250)!(1.3.5\ldots\ldots 499)$$
$$\ldots\ldots\quad\ldots\ldots\quad\ldots\ldots\quad\ldots\ldots$$

Proceeding thus we find the powers required are

$$2^{994},\quad 3^{498},\quad 5^{249},\quad 7^{164}.$$

II. Solution by DR. O. E. GLENN, Drury College.

The theorem covering the general problem is due to Gauss,* and is the following: If p is any prime less than or equal to m, then the highest power of p dividing $m!$ is

$$p^{[m/p]+[m/p^2]+[m/p^3]+\cdots} = p^{\sum_{i=1}^{\infty}[m/p^i]}$$

where $[s/t]$ stands for the greatest integer in s/t. Applying this we have $2^{500+250+125+62+31+15+7+3+1}=2^{994}$, and similarly for the others.

Also solved by A. H. Holmes, and G. B. M. Zerr.

GEOMETRY.

263. Proposed by FREDERICK R. HONEY, Trinity College, Hartford, Conn.

Construct a sphere whose surface shall intersect the surface of any four given spheres in great circles.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Let P be the center of a circle intersecting two circles, centers C, C', in the extremities of diameters AB, $A'B'$, respectively. Draw through P a perpendicular to CC', intersecting it in D. Then, if r be the radius of the intersecting circle, we have

$$r^2 = PC^2 + CA^2 = PC'^2 + C'A'^2.$$

$$\therefore PD^2 + DC^2 + CA^2 = PD^2 + DC'^2 + C'A'^2, \text{ and } DC^2 - DC'^2 = C'A'^2 - CA^2.$$

Hence D is a fixed point, and the locus of P is consequently a fixed line. By rotating the figure about CC' we find that the locus of the center of a sphere intersecting two given spheres in great circles is a certain plane.

Constructing these planes for three pairs of the given spheres, each sphere being involved, we get a common point as the center of the required sphere, assuming that the centers of the given spheres are not coplanar.

264. Proposed by B. F. FINKEL, A. M., Drury College, Springfield, Mo.

Let l and m be two straight lines intersecting in A. With A as center and any radius r describe a circle intersecting l and m in E, M and G, Q, respectively, and the bisector of the opposite angles formed by l and m in F and K. With I, the middle point of EA, as center, and radius, r, describe an arc intersecting the bisector of the opposite angles formed by l and m in O. With O as center, and radius $OA+r$ describe circle $FHCDBJF$; F and D the points of intersection of this circle with the bisector of opposite angle; H, B the intersections on l, and J, C on m. What is the ratio of arc HFJ to arc BD?

*Disquisitiones Arithmeticae.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let 2θ be the angle formed by l, m. Also let A be the origin. Then ($\frac{1}{2}r\cos\theta$, $\frac{1}{2}r\sin\theta$) are the coördinates of I where FD is the axis of abscissas.

$\therefore x^2+y^2-rx\cos\theta-ry\sin\theta=\frac{3}{4}r^2$ is the equation to the circle, center I.

$\therefore y^2-ry\sin\theta=\frac{3}{4}r^2$ or

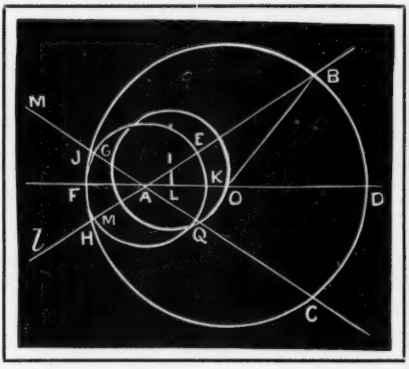
$y=\frac{1}{2}r\sin\theta\pm\frac{1}{2}\sqrt{(3r^2+r^2\sin^2\theta)}$.

$\therefore OD=OA+r=y+r=r[1+\frac{1}{2}\sin\theta+\frac{1}{2}\sqrt{(3+\sin^2\theta)}]$.

Then if $\angle BOD=\phi$,

$$AO:BO=\sin(\phi-\theta):\sin\theta.$$

$$\therefore \sin(\phi-\theta)=\frac{[\sin\theta+\sqrt{(3+\sin^2\theta)}]\sin\theta}{2+\sin\theta+\sqrt{(3+\sin^2\theta)}}$$



This gives ϕ. Now arcHFJ:arc$BD=2\theta r:\phi DO$.

$\therefore$ arcHFJ:arc$BD=2\theta:[2+\sin\theta+\sqrt{(3+\sin^2\theta)}]\phi$.

265. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Find the Cartesian equation of a curve in a vertical plane such that a particle, sliding down the curve under the force of gravity alone, will require to pass from any point of beginning to the lowest point of the curve, a time proportional to the square of the distance to be traversed along the curve.

Solution by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

With the usual notation, the equation of motion is

$$\frac{ds^2}{dt^2}=2gy \quad \ldots\ldots (1).$$

Let $t=ks^2$, or $s\sqrt{k}=\sqrt{t}$ ..... (2). This gives

$$\frac{d^2s}{dt^2}=\frac{1}{4kt}=\frac{1}{4k^2s^2}=2gy \quad \ldots\ldots (3), \text{ or, } s=\frac{1}{2k\sqrt{2g}\sqrt{y}} \quad \ldots\ldots (4).$$

Differentiating, $ds=-\frac{1}{4k\sqrt{2g}}\cdot\frac{1}{y^{\frac{3}{2}}}$ ..... (5). But $dx^2+dy^2=ds^2$ ..... (6).

Substituting (5) in (6), and arranging,

$$dx=\frac{\sqrt{(1-16k^2gy^3)}}{4k\sqrt{g}\,y^{\frac{3}{2}}}dy,$$

the differential equation of the curve, cartesian coördinates.

Also solved by G. B. M. Zerr. Professor Greenwood derives the *intrinsic* equation in the simple form $k^2=4s^2g\cos\psi$.

266. Proposed by DR. O. E. GLENN, Drury College.

Given the feet of the three perpendiculars from any point a on the circumcircle to the sides of the triangle are collinear, then if on the three chords $\overline{a\,b}$, $\overline{a\,c}$, $\overline{a\,d}$, as diameters circles be described, the points of intersection of these circles are collinear. [Salmon's *Higher Plane Curves*].

I. Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Let A, B, and C be the points of intersection of the circumferences described on ab, ac, and ad as diameters. Then since the diameters are concurrent at a, and A is common to two circles which have a common chord, $\angle cAa$ and $\angle dAa$ are right angles, and therefore their two non-coincident sides form a straight line through c and d. Likewise B is collinear with b and d, and consequently with c and d. Also C is collinear with b and c since there cannot be more than one perpendicular to the common chord at C. Therefore A, B, and C are collinear.

II. Solution by the PROPOSER.

By inversion with respect to a as a center, ab, ac, ad remain invariant. The given circle inverts into its radical axis with the circle of inversion, and the three circles in question into a triangle in the circle of inversion. The vertices of this triangle lie on a circle passing through a, for perpendiculars from a on the sides are collinear. Hence the invert of a circle through the intersections of the circles in question is a circle through a, and hence the circle through these intersections is a straight line.

Also solved by G. B. M. Zerr.

GROUP THEORY.

10. Proposed by O. E. GLENN, Ph. D.

Find the order of the group of isomorphisms (H) of the group (G) of order p^4 defined by the relations $P_1{}^{p^2}=P_2{}^{p^2}=I$, $P_1P_2=P_2P_1$.

Solution by the PROPOSER.

In the general isomorphism represented by

$$*J=\begin{pmatrix} P_1, & P_2 \\ P_1{}^xP_2{}^y, & P_1{}^zP_2{}^w \end{pmatrix}$$

the operation $P_1{}^\alpha P_2{}^\beta$ corresponds to $P_1{}^{x'}P_2{}^{y'}$, where

$$\begin{matrix} x' \equiv \alpha x+\beta z \\ y' \equiv \alpha y+\beta w \end{matrix} \pmod{p^2} \quad \ldots\ldots\ldots (\mathrm{I}).$$

Hence H is isomorphic with the congruence group defined by (I) and its order h equals the number of sets of values x, y, z, w for which

*Burnside, *Finite Groups*, p. 222.

$$xw - zy \equiv S \pmod{p^2}; \ (S \text{ not} \equiv 0)$$

provided only that at least one letter of the set (x, y), and one of (z, w) are prime to p. Consider first the case $zy \equiv 0 \pmod{p^2}$. This may happen when

1) $z \equiv 0$, with $y \equiv \Phi(p^2) = p(p-1)$ values.
2) $z \equiv 0$, with $y \equiv p^2 - 1 - \Phi(p^2) = p-1$ values.
3) $z \equiv 0$, $y \equiv 0$.
4) $z \equiv \Phi(p^2) = p(p-1)$ values, with $y \equiv 0$.
5) $z \equiv p^2 - 1 - \Phi(p^2) = p-1$ values, with $y \equiv 0$.
6) $z \equiv p^2 - 1 - \Phi(p^2) = p-1$ values, with $y \equiv p^2 - 1 - \Phi(p^2) \equiv p-1$ values.

Then for

1), x may assume $p^2 - 1$, and w, $p(p-1)$ values.
2), x may assume $p(p-1)$, and w, $p(p-1)$ values.
3), x may assume $p(p-1)$, and w, $p(p-1)$ values.
4), x may assume $p(p-1)$, and w, $p^2 - 1$ values.
5), x may assume $p(p-1)$, and w, $p(p-1)$ values.
6), x may assume $p(p-1)$, and w, $p(p-1)$ values.

Hence there are

$$2(p^2-1)p^2(p-1)^2 + 2p^2(p-1)^3 + 2p^2(p-1)^2 + 2(p^2-1)p^2(p-1)^2 + 2p^2(p-1)^3 + 2p^2(p-1)^4 \quad \text{.....(II)}$$

sets of values with either $xw \equiv 0$ or $zy \equiv 0$, for which S is not $\equiv 0 \pmod{p^2}$.

Next let two of the parameters x, y, z, w, be divisible by p, without making $xw \equiv 0$ or $zy \equiv 0$. These must be (x and z) or (y and w). Let x and z be multiples of p, [$(p-1)^2$ sets of them]. Then to y may be assigned $\Phi(p^2) = p(p-1)$ values, for each one of which there are $^*dv(x; p^2) = p$ values of w which make $S \equiv 0$ and $p(p-1) - p = p^2 - 2p$ values giving a value of S not $\equiv 0$; and similarly, taking y and w, divisible by p. We thus obtain

$$2p^2(p-2)(p-1)^3 \text{ sets of } x, y, z, w, \quad \text{.....(III)}.$$

Next let it be assumed that one and but one parameter x, y, z, w, is divisible by p. Then for *all* the $\Phi(p^2)$ values each, of the remaining three parameters, S is not $\equiv 0$, and since there are four ways of taking one letter divisible by p, and $p-1$ values for that letter, this case gives

$$4p^3(p-1)^4 \text{ additional sets} \quad \text{.....(IV)}.$$

Finally, let all the parameters be prime to p. To each of three of them we may assign any one of $\Phi(p^2) = p(p-1)$ values, and then one value of the remaining one is determined making $S \equiv 0$, so that there are $p(p-1) - 1$ values of

*Mathews, *Theory of Numbers*, p. 11.

this remaining one giving a value of S not$\equiv 0 \pmod p$. Thus are obtained $p^3(p-1)^3(p^2-p-1)$ new sets.....(V).

This exhausts the cases to consider. The sum of II, III, IV, and V is h, the order of the automorph of G. This sum is

$$h=p^3(p-1)^3(p+1)(p^2+p-1).$$

MECHANICS.

184. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

A sphere, radius a, rests between two parallel thin perfectly rough rods A and B in the same horizontal plane at a distance apart equal to $2c$; the sphere is turned about A until its center is very nearly vertically over A; it is then allowed to fall back. Prove that it will rock between A and B if $10c^2<7a^2$; also, that θ_r, the angle through which it will turn after the rth impact is given by the equation $\cos\theta_r=\frac{\sqrt{(a^2-c^2)}}{a}+\frac{a-\sqrt{(a^2-c^2)}}{a}\left(1-\frac{10c^2}{7a^2}\right)^{2r}$.

I. Solution by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

Let ω'=the angular velocity of the sphere about its center at any time t' from the beginning of the first stage of motion; k, k_1 the radii of gyration about the center and a point of the surface, respectively; m=its mass, ω_1, ω_2, etc., the angular velocities about the center just after the first, second, etc., impacts; V'=the linear velocity of the center just before the first impact, $p=(a^2-2c^2)/a$ =the perpendicular from a point of contact upon the direction of V', θ'=the angle the radius makes with the vertical corresponding to ω', and g=the acceleration of gravity. Then for the first impact, the moment of angular momentum is

$$mk_1^2\omega_1=(k^2\omega'+V'p)m \text{.....(1), or, } \omega_1=\frac{k^2\omega'+V'p}{k_1^2}\text{.....(2).}$$

For the first stage of motion,

$$mk_1^2\frac{d^2\theta'}{dt'^2}=mga\sin\theta' \text{.....(3).}$$

Integrating (3), noticing that initially, $d\theta'/dt'=0$, $\cos\theta'=1$,

$$mk_1^2\frac{d\theta'^2}{dt'^2}=2mga(1-\cos\theta') \text{.....(4).}$$

When the first stage of motion is completed,

$$\cos\theta'=\frac{\sqrt{(a^2-c^2)}}{a}\text{, and (4) then is, since } k_1^2=\tfrac{7}{5}a^2 \text{.....(5),}$$

$$\omega'^2 = \frac{10g}{7a}\left(1 - \frac{\sqrt{(a^2-c^2)}}{a}\right) \quad \ldots\ldots (6).$$

$$\therefore V' = a\omega' = a\sqrt{(2ag)}\sqrt{\frac{5}{7a^2}\left(1 - \frac{\sqrt{(a^2-c^2)}}{a}\right)} \quad \ldots\ldots (7).$$

(2) then gives, since $k^2 = \frac{2}{5}a^2$,

$$\omega_1^2 = 2ag\left(1 - \frac{\sqrt{(a^2-c^2)}}{a}\right)\left(1 - \frac{10c^2}{7a^2}\right)^2 \div \tfrac{7}{5}a^2 \quad \ldots\ldots (8).$$

For rocking, the factor $1-(10c^2/7a^2)$ must be positive, or $10c^2 < 7a^2$.

For the second stage of motion, θ_1, t_1 corresponding to θ', t' in the first,

$$mk_1^2 \frac{d^2\theta_1}{dt_1^2} = mga\sin\theta_1 \quad \ldots\ldots (9).$$

Integrating (9) and noticing that when $\frac{d\theta_1}{dt_1} = \omega_1$, $\cos\theta_1 = \frac{\sqrt{(a^2-c^2)}}{a}$,

$$mk_1^2 \frac{d\theta_1^2}{dt_1^2} = 2mg\left[\left(1 - \frac{\sqrt{(a^2-c^2)}}{a}\right)\left(1 - \frac{10c^2}{7a^2}\right)^2 + \sqrt{(a^2-c^2)} - a\cos\theta_1\right] \quad \ldots\ldots (10).$$

When the motion ceases for the first time, $d\theta_1/dt_1 = 0$, and then (10) gives

$$\cos\theta_1 = \frac{\sqrt{(a^2-c^2)}}{a} + \left(\frac{a - \sqrt{(a^2-c^2)}}{a}\right)\left(1 - \frac{10c^2}{7a^2}\right)^2 \quad \ldots\ldots (11).$$

For the backward motion, θ'', t'', ω'' corresponding to θ', t', ω', we have

$$mk_1^2 \frac{d^2\theta''}{dt''^2} = mga\sin\theta'' \quad \ldots\ldots (12).$$

Integrating (12), and noticing that, when $\cos\theta'' = \cos\theta_1$, $d\theta''/dt'' = 0$, we have

$$\tfrac{7}{5}ma^2.\omega''^2 = 2mg\left[a\cos\theta'' - \sqrt{(a^2-c^2)} + \left(1 - \frac{\sqrt{(a^2-c^2)}}{a}\right)\left(1 - \frac{10c^2}{7a^2}\right)^2\right] \quad \ldots\ldots (13).$$

When

$$\cos\theta'' = \frac{\sqrt{(a^2-c^2)}}{a}, \text{ (13) gives } \omega''^2 = 2\left(1 - \frac{\sqrt{(a^2-c^2)}}{a}\right)\left(1 - \frac{10c^2}{7a^2}\right)^2 g \div \tfrac{7}{5}a^2 \quad \ldots\ldots (14).$$

Now displacing ω_1, ω', V' by ω_2, ω'', V'', we find

$$\omega_2^2 = 2ag\left(1 - \frac{\sqrt{(a^2-c^2)}}{a}\right)\left(1 - \frac{10c^2}{7a^2}\right)^4 \div \tfrac{7}{5}a^2 \quad \ldots\ldots (15);$$

and by a process entirely analogous to that commencing with equation (9), we find

$$\cos\theta_2 = \frac{\sqrt{(a^2-c^2)}}{a} + \left(\frac{a-\sqrt{(a^2-c^2)}}{a}\right)\left(1-\frac{10c^2}{7a^2}\right)^4 \quad \ldots\ldots(16),$$

and generally, by inference,

$$\cos\theta_r = \frac{\sqrt{(a^2-c^2)}}{a} + \left(\frac{a-\sqrt{(a^2-c^2)}}{a}\right)\left(1-\frac{10c^2}{7a^2}\right)^{2r} \quad \ldots\ldots(17).$$

I. Solution by the PROPOSER.

Let C be the center of the sphere at the instant of impact, ω_{r-1}, ω_r, the angular velocities of the sphere just before and just after the rth impact, respectively; $\angle ACB = 2\beta$, k^2 = radius of gyration, then $(a^2+k^2)\omega_r = (k^2+a^2\cos 2\beta)\omega_{r-1}$. Now, $\sin\beta = c/a$, $k^2 = \frac{2}{5}a^2$.

$$\therefore \tfrac{7}{5}a^2\omega_r = a^2\left(\tfrac{7}{5}-\frac{2c^2}{a^2}\right)\omega_{r-1}. \quad \therefore \frac{\omega_r}{\omega_{r-1}} = 1-\frac{10c^2}{7a^2}.$$

$\therefore 1-\dfrac{10c^2}{7a^2}$ must always be positive. Let ω = angular velocity just before first impact. By the principle of energy,

$$\tfrac{1}{2}(a^2+k^2)\omega^2 = ga(1-\cos\beta),$$
$$\tfrac{1}{2}(a^2+k^2)\omega_r^2 = ga(\cos\theta_r-\cos\beta).$$

$$\therefore \frac{\omega_r^2}{\omega^2} = \frac{\cos\theta_r-\cos\beta}{1-\cos\beta}. \quad \cos\theta_r = \cos\beta + (1-\cos\beta)\left(\frac{\omega_r}{\omega}\right)^2.$$

$$\cos\beta = \frac{\sqrt{(a^2-c^2)}}{a}, \qquad \frac{\omega_r}{\omega} = \left(1-\frac{10c^2}{a^2}\right)^r.$$

$$\therefore \cos\theta_r = \frac{\sqrt{(a^2-c^2)}}{a} + \frac{a-\sqrt{(a^2-c^2)}}{a}\left(1-\frac{10c^2}{a^2}\right)^{2r}.$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

247. Proposed by PROFESSOR G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Find the sum, to n terms, of

$$1+\frac{n}{2}+\frac{n(n+2)}{2.4}+\frac{n(n+2)(n+4)}{2.4.6}+\ldots\ldots$$

248. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\frac{6435}{2}\cdot\frac{161280^2}{929569}\left[1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}+\ldots\ldots\right]=\pi^{16}$.

249. Proposed by J. J. KEYES, Fogg High School, Nashville, Tenn.

Solve $x+y+z=5$, $x^2+y^2=z^2$, $x^3+y^3+z^3=8$.

AVERAGE AND PROBABILITY.

173. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A chord of length c is drawn at random in a given ellipse. What is the average area of the segment cut off by the chord?

CALCULUS.

209. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A thread makes $n(=30)$ equidistant spiral turns around a rough cone whose altitude is $h(=10$ feet), and radius of base $r(=11$ inches). How far will a bird fly in unwinding the thread if the part unwound is at all times perpendicular to the axis of the cone?

210. Proposed by EDWIN L. RICH, Schenectady, New York.

Determine a polynomial, $f(x)$, entirely in x and of the seventh degree, so that $f(x)+1$ is divisible by $(x-1)^4$ and $f(x)-1$ by $(x+1)^4$. [Goursat-Hedrick, *A Course in Mathematical Analysis*, p. 32, Ex. 3.]

211. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

If $x=v^{1/(v-1)}$, what is the $f(x)$ such that $v=f(x)$?

DIOPHANTINE ANALYSIS.

130. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

In how many ways may a number of which the prime factors are known, be expressed as the sum of two or more consecutive numbers?

131. Proposed by DR. O. E. GLENN, Drury College.

Devise a method of finding the cubic residues of a number, analogous to Gauss's "Method of Exclusion" for quadratic residues.

GEOMETRY.

274. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

If a straight line AB is placed between two intersecting straight lines MN and PQ and is made to revolve through all possible positions having A always in MN and B always in PQ, what is the locus of any point L in AB or AB produced?

275. Proposed by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

An hyperbola is drawn touching the axes of an ellipse, and the asymptotes of the hyperbola touch the ellipse. Prove that the center of the hyperbola lies on one of the equal conjugate diameters of the ellipse.

276. Proposed by G. I. HOPKINS, Manchester, N. H.

ABC is an equilateral triangle whose vertices are the centers of circles with radius AB, and H is the center of the arc AB. From F, the point of intersection of the circles whose centers are A and C, a line is drawn through H to the circumference CAN. Draw BN, and prove that the angle ABN is an angle of a regular pentagon.

MECHANICS.

186. Proposed by R. D. CARMICHAEL, Hartselle, Alabama.

A point P keeps at uniform distance from and moves with uniform angular velocity around a point Q which is in harmonic motion, making one revolution while Q swings to and fro. If P is in the line of the path of Q and on the same side of the center of that path with Q when Q is at the extremity of the path, what is the locus of P?

UNSOLVED PROBLEMS.

NOTE. The following problems still remain unsolved (in our columns).

Algebra, 179. Proposed by DR. L. E. DICKSON, The University of Chicago.

Find the roots of the algebraically solvable quintic equation

$$x^5+qx^2+px+\tfrac{1}{5}\left[\frac{q^2}{p}-\frac{p^3}{5q}\right]=0.$$

Geometry, 267. Proposed by W. W. LANDIS, Dickinson College, Carlisle, Pa.

Prove that every orthogonal system of circles is an isothermal system.

Group Theory, 9. Proposed by DR. L. E. DICKSON, The University of Chicago.

Does there exist a triply transitive group on m letters of order $m(m-1)(m-2)$ other than the linear fractional group in the Galois Field of order $p^n=m-1$ and the group 720_3 on ten letters (Cole, *Quarterly Journal*, 1895, p. 44)? This question relates to Problem 99, MONTHLY, March, 1900.

Miscellaneous, 151. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Sum the series $\sum_{r=1}^{r=m} \operatorname{cosec}\left[\frac{2r-1}{4m}\pi+\theta\right]\operatorname{cosec}\left[\frac{2r-1}{4m}\pi-\theta\right]$.

NOTES AND NEWS.

Mr. A. R. Maxson has been appointed tutor in mathematics at Columbia University.

Mr G. H. Light has been appointed instructor in mathematics at Purdue University.

Purdue University has appointed Mr. W. A. Tehring as instructor in pure mathematics.

Mr. Arthur Ranum has been appointed assistant in mathematics at Stanford University.

Miss Florence Hanington has been appointed fellow in mathematics at Bryn Mawr College.

Mr. H. W. Stager and Mr. T. G. Brown are assistants in mathematics at Stanford University.

Mr. W. A. Pickering has received the appointment of professor of applied mathematics at Cardiff.

A technical high school with accommodations for 1200 people will be erected at Newark, N. J.

Miss Louise Duffield Cummings has been appointed scholar in mathematics at Bryn Mawr College.

Mr. Paul N. Peck has been appointed instructor in mathematics in the George Washington University.

Mr. Edwin Haviland, A. M. (Cornell), is a candidate for the doctorate in mathematics at Stanford University.

At Grinnell College, Assistant Professor W. J. Rusk has been promoted to a full professorship in mathematics.

Mr. G. M. Conwell has received the appointment of fellow in mathematics at Princeton University for the year 1905-1906.

At the College of the City of New York Mr. G. C. Daly, Mr. H. W. Powell, and Mr. L. P. Siceloff have been appointed tutors in mathematics.

The Bolyai prize has been awarded to M. Poincaré, in recognition of his researches in mathematics during the past five years. The awarding commission met in Budapest last month.

There will be a Civil Service examination on December 6 to fill a vacancy in the position of computer (male) in the forest service. The salary attached to the position is $1000 per annum.

Professor Arnold Emch, formerly of the University of Colorado, has assumed his new duties as professor of mathematics at his alma mater, the Cantonal College of Solothurn, Switzerland.

On Tuesday, January 9, 1906, Dr. George Bruce Halsted will deliver an address at Ohio State University, under the auspices of the Philosophical Society on "The Non-Euclidean Contribution to Philosophy."

The Academy of Sciences of Berlin held its Leibnitz session on June 29. The sum of six thousand marks was set apart in recognition of the investigations of the late Guido Hauck, but the Steiner prize was not awarded.

Dr. Arthur G. Hall, formerly instructor in mathematics at the University of Michigan, and associate professor of mathematics at the University of Illinois, has been called to the professorship of mathematics at Miami University.

Professor C. J. Keyser is in editorial charge of the mathematical department of the Encyclopedia Americana, about to be issued by the *Scientific American*. Special attention has been given to this department, and over forty articles have been written for the Encyclopedia by American mathematicians on their several special fields of investigation.

The following have been elected members of the American Mathematical Society: Lieutenant Colonel C. P. Echols, U. S. Military Academy; Professor G. B. Guccia, University of Palermo; Professor H. B. Evans, University of Pennsylvania; Dr. A. M. Hiltebeitel, Princeton University; Dr. J. M. Poor, Dartmouth College; Professor J. E. Williams, Virginia Polytechnic Institute.

The London Mathematical Society held its annual general meeting at the Registered Office of the Society, 22 Albemarle Street, London, on November 9th. The following officers were elected for the ensuing year: President, A. R. Forsyth, Sc. D., F. R. S; Vice-Presidents, W. F. Burnside, Sc. D., F. R. S., Sir William Niven, K. C. B., F. R. S.; Treasurer, J. Larmor, D. Sc., F. R. S.; Secretaries, A. E. H. Love, D. Sc., F. R. S., and J. H. Grace, M. A.

The American Mathematical Society held its October meeting at Columbia University on Saturday, October 28th. The program issued for the meeting announced the following papers: "On the Cayley-Veronese Class of Configurations," by Professor W. B. Carver; "Multiple Improper Integrals," by Professor James Pierpont; "Poncelet Quadrilaterals on a Curve of the Third Order and a Conic," by Professor H. S. White; "On the Geodesics Passing through a Given Point of a Surface," by Dr. Edward Kasner.

The November number of the *Bulletin of the American Mathematical Society* contains the following papers: "A Set of Generators for Ternary Linear Groups," by Ida May Schottenfels; "Note on the Structure of Hyper-Complex Number Systems," by Saul Epsteen; "A Geometric Property of the Trajectories of Dynamics," by Edward Kasner; "On the Possible Numbers of Operators of

Order 2 in a Group of Order 2^m," by G. A. Miller; "On the Arithmetic Nature of the Coefficients in Groups of Finite Monomial Linear Substitutions," by W. A. Manning.

The fifth annual meeting of the Central Association of Science and Mathematics Teachers will be held in Chicago (153 LaSalle Street) on November 30, December 1 and 2. The following addresses will be made before the mathematics section: "The Straight Line in Geometry," by J. W. Withers, Teachers College, St. Louis; "Interest and Progress in the Teaching of Mathematics," by N. J. Lennes, Wendell Phillips High School, Chicago; "Aims in Teaching Algebra," by Professor Robert J. Aley, Indiana University; "Some Thoughts on the Teaching of Geometry," by C. A. Patterson.

Rev. J Edward Kirbye, D. D., was inaugurated as the fourth president of Drury College on November 9th. The installation ceremonies were signalized by a special reunion of the alumni of the college and its friends. "Democracy and Higher Education" was the subject of President Kirbye's inaugural address. Other educators who delivered addresses were Rev. Stephen M. Newman of Washington, D. C., Chancellor W. S. Chaplin of Washington University, Professor John Picard of the University of Missouri, Professor George E. Comstock of the University of Wisconsin, Father Roger of St. Louis University.

At the last commencement at Grinnell College, Professor Samuel J. Buck retired from active service and was made professor emeritus after a continuous service of forty-one years at that institution. He graduated from Oberlin College in 1858. In 1862, he graduated from Oberlin Theological Seminary, entered the University in 1864, was made principal of the Academy in Grinnell College in 1864, and made professor of mathematics and physics in the college in 1869. In 1893 he was elected professor of mathematics and astronomy at the same college, which position he has held up to the present time. July fourth, 1905, Professor Buck was seventy years old. He is in excellent health and leaves the honored position which he has made and brought to high college rank after more than two scores of years of service, appreciation for which was shown in marked way by the friends of the college at the time of his retirement.

Princeton University has, during the past year, adopted and put into practice the preceptorial system of instruction, largely increasing her staff of professors in all departments. As a result, the following changes have gone into effect in the mathematical department this Fall: Mr. J. H. Jeans, of Trinity College, Cambridge has been appointed professor of applied mathematics to fill the chair vacated by Prof. E. O. Lovett on his being transferred to the department of astronomy. Dr. L. P. Eisenhart and Dr. William Gillespie have been promoted to preceptorships and the following have been called from other institutions to preceptorships: Dr. G. A. Bliss from the University of Missouri, Dr. Oswald Veblen from the University of Chicago, Professor John W. Young from the Northwestern University. The newly appointed instructors are: C. R. MacInnes from the Johns Hopkins University, A. L. Underhill and E. B. Morrow.

A curious poetical tribute to Archimedes, composed by a French mathematician, is recalled by *The Academy* (London). The first calculation of the numerical value of π is ascribed to Archimedes, and, carried out to thirty decimal places, this value is 3.14159265358979323846264338327 9. Each of the thirty-one words in the quatrain, in order, contains the number of letters given by the corresponding figure in the numerical value of π, as follows:

Que j' aime á faire apprendre un nombre utile aux sages
Immortel Archimede, artiste ingénieur!
Qui de ton jugement peut priser la valeur
Pour moi ton problème eut de pareils avantages.

The *Frankfurter Zeitung* adds the following similar "effort" from a German "poet and geometrician:"

Dir o Held, o alter Philosoph, Du Riesen-Genie!
Wie viele Tausende bewundern Geister himmlisch wie Du und göttlich!
Noch reiner in Aeonen wird das uns strahlen wie im lichten Morgenrot!

The translation of the French is: "How I love to teach a number useful to the wise immortal Archimedes, artist—engineer! Who can appraise the worth of thy judgment? For me thy problem has equal advantages." The German runs thus: "To thee, O hero, O old philosopher, O giant genius! How many thousand souls wonder, heavenly as thou and divine! Yet clearer in the ages will that stream on us than in the luminous dawn." Neither of these can be said to be very clear, but considering the limitations of their composition they are certainly remarkable, gravely comments the *Literary Digest*. Meanwhile, will some one give us an English version of π?

BOOKS AND PERIODICALS.

Advanced Algebra. By Herbert E. Hawkes, Ph. D., Assistant Professor of Mathematics in Yale University. Cloth, 8vo., XIV + 285 pages. List price, $1.40; mailing price, $1.50. Ginn & Co., Boston, New York, Chicago, London.

An algebra preëminently suited for short college courses should be comprehensive enough to set forth amply all the parts of the science that are recognized as essentials, and it should be terse enough to be covered in a college term without skipping chapters and parts of chapters, and thus interfering with the continuity of the student's development. *Advanced Algebra* is a book of decided merit when judged from this standpoint. The parts of the science usually taught in college freshman algebra courses are presented in less than three hundred pages, and the treatment is, on the whole, exceptionally clear and forceful. The text is full of the laboratory spirit; graphical representations, and graphical methods of solution of equations form an integral part of Professor Hawkes' work, and they are

introduced early in the text. Necessarily the brevity of treatment has curtailed fundamental and even indispensable general theory, but the book is primarily aimed to "present in concise but clear form the portions of algebra that are required for entrance to the most exacting universities and technical schools," and it will supply a definite want in this field. G.

The Essentials of Algebra. By Robert J. Aley, Ph. D., and David A. Rothrock, Ph. D., Professors of Mathematics at Indiana University. Chapter XX, Logarithms; 15 pages. Silver, Burdett & Co., New York, Boston, Chicago.

This chapter is bound with the new edition of this successful and in many respects epoch-marking secondary school algebra, at the request of a number of friends of the book who wish to teach the theory of logarithms in their high school courses. The chapter is also to be had in pamphlet form. The main theorems of the subject are developed and a five place table for argument one to one hundred given. The principles find application in a treatment of compound interest and annuities. G.

Elements of Descriptive Geometry. By O. E. Randall, Ph. D,, Professor of Mechanical Drawing, Brown University. 8vo. Semi-flexible cloth. IV+209 pages. List price, $2.00; mailing price, $2.10. Ginn & Co., Boston, New York, Chicago, London.

This work is a text book for colleges and engineering schools. The aim otfhe treatise is to make a clear presentation of the theory of projection and to show its application as a medium of expression. By the discussion and proof of a great variety of problems the author aims to enable the student to make a ready and intelligent use of this medium in the representation of all forms of magnitudes. Since by far the greater part of practical drafting is done from the standpoint of the third quadrant, the principles of descriptive geometry are presented from the standpoint of the same quadrant. It seems to us that Professor Randall's treatise is one of the very best of a beautiful line of texts in this subject which has appeared in the past few years. The principles of the subject are here classified and pedagogically arranged, and are illustrated by numerous figures and drawings. In fact, this is the first treatise on the subject that has come to our notice which makes the illustrative drawing do its full part. Some special features will commend themselves to teachers of mechanical and descriptive drawing:

Free use is made of profile and other supplementary planes of projection. Isometric projection and other forms of one-plane projection are treated as applications of descriptive geometry. A method of locating given parts is incorporated. As a result the work of the drawing and recitation room may be easily and definitely assigned. G.

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The closing (October) number of Volume 6 of the *Transactions of the American Mathematical Society* contains the following papers: Maurice Fréchet, "Sur l'écart de deux courbes et sur les courbes limites:" John Eisland, "On a Certain System of Conjugate Lines on a Surface Connected with Euler's Transformation;" L. P. Eisenhart, "Surfaces of constant Curvature and their Transformations;" N. J. Lennes, "Volumes and Areas;" E. O. Lovett, "On a Problem Including that of Several Bodies and Admitting of an Additional Integral;" F. R. Sharpe, "On the Stability of the Motion of a Viscous Liquid;" A. Loewy, "Ueber die Vollständig Reducibeln Gruppen, die zu einer Gruppe Linearer Homogener Substitutionen Gehören;" W. B. Carver, "On the Cayley-Veronese Class of Configurations." This number also contains; Notes and Errata, Volumes 5, 6; Table of Contents, Volume 6.

12

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EXPRESSIONS FOR THE ELEMENTS OF A DETERMINANT IN TERMS OF THE MINORS OF A GIVEN ORDER. GENERALIZATION OF A THEOREM DUE TO STUDNICKA.

By L. E. DICKSON.

I. It is desired to express the n^2 elements a_{ij} of a determinant $D \equiv |\, a_{ij} \,|$ in terms of the $C^2_{n,m}$ minors of order m. Arrange these minors in a determinant Δ so that there shall appear in one row [column] all the $C_{n,m}$ minors built from the elements of the same m rows [columns] of D; then Δ is called the mth compound of D. It is well known that

$$D^{C_{n-1,\ m-1}} = \Delta. \tag{1}$$

For the applications in view it suffices to assume that $\Delta \neq 0$, whence $D \neq 0$.

One symmetrical method of treating the problem consists in determining all the minors M_{m-1} of order $m-1$, then all the minors M_{m-2}, ..., and finally the a_{ij}. We first determine D from (1) by a root extraction.* By Studnicka's

*We can however find D by the extraction of a dth root, where d is the greatest common divisor of m and n, since D^d equals a sum of products of the M_m. For instance, if $d=1$, this follows from Laplace's expansion of D in the form $\Sigma \neq M'_m M''_m \ldots\ldots M_m^{(n/m)}$, where M'_m is formed from the first m rows of D, M''_m from the second m rows, etc. Again, if $d=2$ take $n=6$, $m=4$ for brevity of writing; then